# Tiebreak proposals for Swiss tournaments

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Discussions about tiebreaks goes back to as far as 1873 (Ahrends, p. 184). Vivid discussions arose when the famous 1914 Mannheim chess tournament was interrupted by World War One. At that point Alekhine was a half point ahead. But Vidmar did have met stronger opponents. The issue of defining rankings for tournaments has been studied in various fields like sports, psychology, decision theory, Condorcet voting. Many ranking methods based on different motivations have been defined.

## Direct methods

### Solkoff, Buchholz, and variants

The above methods have a practical disadvantage. If a player leaves the tournament early the Solkoff contribution will be flatted in a negative sense. His opponents are thereby disadvantaged. The FIDE has found a solution for this (Forlani). But this solution is complicated and artificial. All direct methods have a fundamental limitation. They do take into account the direct opponents, but not the opponents of the opponents. And the opponents thereof. Four alternative indirect tiebreak methods are presented.

## Relative Elo Ratings

#### The normal distribution function

"From general experience in sports we know that the stronger player does not invariably outperform the weaker. A player has good days and bad, good tournaments and bad. By and large at any point in his career, a player will perform around some average level. Deviations from this level occur, large deviations less frequently than small ones. These facts suggest the basic assumption of the Elo system. It is best stated in the formal terms of statistics:"

> The many performances of an individual wil be normally distributed, when evaluated on an appropriate scale (Elo, 1978, p19).

#### Development of the Percentage Expectancy Table

The normal probabilities may be taken directly from the standard tables of the areas under de normal curve, when de difference D in ratings is expressed as a z score. The z value of a difference equals D / sigma, where sigma = 2000 / 7. For example, let D = 160. Then z = 160 * 7 /2000 = 0,56. The table / formula gives ,7123 and ,2877 as areas of the two portions under de cumulative curve. These numbers represents the expected scores of the two opponents (Elo, p. 158).

In a rating system (KNDB, FMJD) ratings are updated at regular time intervals. Relevant games are considered. Game by game the difference between actual and expected score is established, according to the percentage expectancy function. Then the existing rating is updated, proportional to the total difference between expected and actual score. However, when computing the relative Elo ratings, we do not assume any existing ratings prior to the tournament. All players in the tournament are assigned a rating, the relative Elo rating, such that for each player the expected score and the actual score are equal. This rating is calculated by a numeric iteration procedure, as no closed formula exists. The relative Elo rating of a tournament is unique, up to a constant.

**Example Open de Sangmelima Cameroon 2014**

At the start of the iterations (R0) one assumes all players have equal strength. In the subsequent iterations the rating of each player is updated according to -ln(We/W) * 2000 / 7. "We" is the expected score derived from the cumulative normal distribution, W is the actual score and ln() constitutes the natural logarithm function. The updates are normalized to set the sum of all updates to zero. The iteration is slow, but stable. We may choose at start any average rating level including zero. This does not effect the ordering of the players.

| Pl | Title | Name | Cn | Rating | + | = | - | N | Pts | Whl | Wl | R0 | We0 | Up0 | R1 | We1 | Up1 | R2 | We2 | Up2 | Rk | R28 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | gmi | Leopold Kouomou | cm | 2296 | 3 | 3 | 0 | 6 | 9 | 25 | 34 | 0,0 | 6,0 | 137,7 | 137,7 | 8,0 | 32,6 | 170,3 | 8,1 | 36,2 | 2 | 304,0 |
| 2 | | Gerard Ngankou | cm | 2265 | 3 | 3 | 0 | 6 | 9 | 25 | 34 | 0,0 | 6,0 | 137,7 | 137,7 | 7,7 | 42,9 | 180,6 | 8,0 | 38,3 | 1 | 312,7 |
| 3 | | Mouanji Iliassou | cm | | 4 | 0 | 2 | 6 | 8 | 16 | 23 | 0,0 | 6,0 | 104,0 | 104,0 | 9,0 | -33,9 | 70,1 | 8,5 | -10,6 | 6 | 18,3 |
| 4 | | Armand Abouem | cm | | 3 | 1 | 2 | 6 | 7 | 30 | 39 | 0,0 | 6,0 | 65,9 | 65,9 | 6,0 | 41,7 | 107,6 | 6,5 | 25,6 | 3 | 185,3 |
| 5 | | Landry Nga | cm | | 1 | 5 | 0 | 6 | 7 | 28 | 37 | 0,0 | 6,0 | 65,9 | 65,9 | 6,3 | 31,1 | 97,0 | 6,6 | 20,5 | 4 | 180,9 |
| 6 | mi | Bruno Fopa | cm | 2262 | 2 | 3 | 1 | 6 | 7 | 21 | 28 | 0,0 | 6,0 | 65,9 | 65,9 | 7,7 | -28,3 | 37,5 | 7,5 | -12,8 | 9 | -26,6 |
| 7 | | Desire Ghuendou | cm | | 2 | 3 | 1 | 6 | 7 | 18 | 25 | 0,0 | 6,0 | 65,9 | 65,9 | 8,2 | -45,6 | 20,3 | 7,6 | -17,6 | 10 | -63,0 |
| 8 | | Maturin Tomi | be | | 2 | 2 | 2 | 6 | 6 | 26 | 35 | 0,0 | 6,0 | 21,8 | 21,8 | 6,4 | -17,1 | 4,7 | 6,0 | 6,0 | 5 | 33,6 |
| 9 | | Arnaud Foto | cm | | 2 | 2 | 2 | 6 | 6 | 25 | 32 | 0,0 | 6,0 | 21,8 | 21,8 | 6,3 | -15,9 | 6,0 | 6,2 | -3,6 | 8 | -19,4 |
| 10 | | Patrick Akono | cm | | 1 | 3 | 2 | 6 | 5 | 31 | 40 | 0,0 | 6,0 | -30,3 | -30,3 | 4,4 | 36,2 | 5,9 | 5,2 | -5,6 | 7 | -16,3 |
| 11 | | Bernard Mambo | cm | | 2 | 0 | 4 | 6 | 4 | 28 | 37 | 0,0 | 6,0 | -94,0 | -94,0 | 3,7 | 18,8 | -75,3 | 4,2 | -8,6 | 11 | 102,5 |
| 12 | | David Wabo Daco | cm | | 1 | 2 | 3 | 6 | 4 | 23 | 31 | 0,0 | 6,0 | -94,0 | -94,0 | 4,7 | -45,7 | -139,7 | 4,1 | -3,3 | 12 | 177,2 |
| 13 | | Barel Bimogo | cm | | 1 | 1 | 4 | 6 | 3 | 23 | 31 | 0,0 | 6,0 | -176,2 | -176,2 | 3,8 | -66,9 | -243,1 | 3,1 | -5,4 | 13 | 299,4 |
| 14 | | Prince Mvondo | cm | | 1 | 0 | 5 | 6 | 2 | 28 | 37 | 0,0 | 6,0 | -292,1 | -292,1 | 1,7 | 50,2 | -241,8 | 2,5 | -59,1 | 14 | 330,6 |

It is easy to check the result of the above calculation:

| Gerard Nganko | 313,0 | | |
|---|---|---|---|
| Opponents | R28 | Diff | We |
| Patrick Akono | -16,3 | 329,3 | 1,75 |
| Landry Nga | 180,9 | 132,1 | 1,36 |
| Maturin Nyamsi Tomi | 33,6 | 279,4 | 1,67 |
| Bernard Mambo | -102,5 | 415,5 | 1,85 |
| Leopold Kouogueu Kouomou | 304,0 | 9,0 | 1,03 |
| Armand Abouem | 185,3 | 127,7 | 1,35 + |
| Expected Score | | | 9,0 points |

| Leopold Kouogueu Kouomou | 304,0 | | |
|---|---|---|---|
| Opponents | R28 | Diff | We |
| Prince Arnaud Mvondo | -330,6 | 634,6 | 1,97 |
| Maturin Nyamsi Tomi | 33,6 | 270,4 | 1,66 |
| Patrick Akono | -16,3 | 320,3 | 1,74 |
| Armand Abouem | 185,3 | 118,7 | 1,32 |
| Gerard Ngankou | 312,7 | -8,7 | 0,98 |
| Landry Nga | 180,9 | 123,1 | 1,33 + |
| Expected score | | | 9,0 points |

## Domain

The iteration requires an indivisible (that is, strongly connected or irreducible) domain. In every possible partition of players into two nonempty subsets, some player of the second set has defeated at least one player in the first set (Glickman, p. 5). Finding all maximal strongly connected components is a mathematical puzzle in its own right.

Remark: one can introduce a virtual opponent equivalent to all other players to make the domain indivisible.

# Least Squares Method

## Principle

If a player has a win, loss or draw his comparison outcome is set to 1, -1 or 0 respectively. The least-squares method constructs a mean square approximation of the skew symmetric comparison outcomes R by the differences of the desired least-squares ratings (Cheboratev, 1999, p. 18):

> Determine $q = (q_1,...,q_n)$ such that $\Sigma_i\Sigma_j(r_{ij} - (q_i-q_j)n_{ij})^2$ is a minimum (Gulliksen, 1956)

where

- $n_{ij}$ is the number of games between player i and j, i ≠ j, zero otherwise
- $r_{ij}$ equals the sum of the results of player i against j

## Calculation

The LSM-ratings (q) are a solution of the set of linear equations:

- L.q = s
- Σq = 0

where

- $L = (l_{ij})$ is the Laplacian matrix of the game results graph
- $l_{ii}$ = Number of comparisons of player i
- $l_{ij}$ = Negative the number of games between player i and j, i ≠ j
- $s = (s_i)$, $s_i = \Sigma_k r_{ik}$, row sum of wins minus losses

**Note** that if the percentage expectancy function of the relative Elo ratings is replaced by a linear relationship, for example the "algorithm of 400", then relative Elo ratings are identical to LS-ratings except for a scaling factor. Multiplying LS-ratings with a factor 800 gives a comparable rating in the Elo-domain (FIDE Rating Regulations effective from 1 July 2017, table 8.1b).

## Example Open de Sangmelima Cameroon 2014

| Pl | Title | Name | Cn | Rating | + | N | s | Whl | Wl | Rk | LS-Rtg |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | gmi | Leopold Kouomou | cm | 2296 | 3 | 6 | 3 | 25 | 34 | 2 | 0,643 |
| 2 | | Gerard Ngankou | cm | 2265 | 3 | 6 | 3 | 25 | 34 | 1 | 0,708 |
| 3 | | Mouanji Iliassou | cm | | 4 | 6 | 2 | 16 | 23 | 5 | 0,059 |
| 4 | | Armand Abouem | cm | | 3 | 6 | 1 | 30 | 39 | 3 | 0,420 |
| 5 | | Landry Nga | cm | | 1 | 6 | 1 | 28 | 37 | 4 | 0,392 |
| 6 | mi | Bruno Fopa | cm | 2262 | 2 | 6 | 1 | 21 | 28 | 9 | -0,042 |
| 7 | | Desire Ghuendou | cm | | 2 | 6 | 1 | 18 | 25 | 11 | -0,122 |
| 8 | | Maturin Tomi | be | | 2 | 6 | 0 | 26 | 35 | 6 | 0,054 |
| 9 | | Arnaud Foto | cm | | 2 | 6 | 0 | 25 | 32 | 10 | -0,047 |
| 10 | | Patrick Akono | cm | | 1 | 6 | -1 | 31 | 40 | 8 | -0,030 |
| 11 | | Bernard Mambo | cm | | 2 | 6 | -2 | 28 | 37 | 12 | -0,234 |
| 12 | | David Wabo Daco | cm | | 1 | 6 | -2 | 23 | 31 | 13 | -0,425 |
| 13 | | Barel Bimogo | cm | | 1 | 6 | -3 | 23 | 31 | 15 | -0,694 |
| 14 | | Prince Mvondo | cm | | 1 | 6 | -4 | 28 | 37 | 14 | -0,683 |

It is easy to check the result of the above calculation as follows:

| Gerard Ngankou | 0,708 | 3 |
|---|---|---|
| Opponents | LS-Rtg | Diff |
| Patrick Akono | -0,030 | 0,738 |
| Landry Nga | 0,392 | 0,316 |
| Maturin Nyamsi Tomi | 0,054 | 0,654 |
| Bernard Mambo | -0,234 | 0,942 |
| Leopold Kouogueu Kouomou | 0,643 | 0,065 |
| Armand Abouem | 0,420 | 0,288 |
| s1 | | 3,00 (Wins minus losses) |

| Leopold Kouogueu Kouomou | 0,643 | 3 |
|---|---|---|
| Opponents | LS-Rtg | Diff |
| Prince Arnaud Mvondo | -0,683 | 1,326 |
| Maturin Nyamsi Tomi | 0,054 | 0,589 |
| Patrick Akono | -0,030 | 0,673 |
| Armand Abouem | 0,420 | 0,223 |
| Gerard Ngankou | 0,708 | -0,065 |
| Landry Nga | 0,392 | 0,251 + |
| s2 | | 3,00 (Wins minus losses) |

# Recursive Buchholz

Let

Rb - Recursive Buchholz ratings
RbOpp - Average Opponents Rb-rating
s - Score percentage - 50%, or alternatively,
- Average of the skew symmetric score points, or
- Average of Wins -/- Losses

Rb, RbOpp, s are n×1 column vectors indexed by player. The Recursive Buchholz Rating Rb is a solution of: (Gonzalez-Diaz et al., 2011, p. 7)

- Rb = RbOpp + s
- ΣRb= 0

Remark: although the perspective of RB and LSM are quite different the resulting order is equivalent

## Skew symmetric score systems

A score system is skew symmetric if for all possible outcomes (x, y),

- x + y = 0

Any traditional score system can be transformed to this form (Chebotarev, 1989, p.1104):

```
    |x 3 .|                   |x 0 .|
A = |0 x 1|    transpose(A) = |3 x 1|
    |. 1 x|                   |. 1 x|

R = (A - transpose(A)) / 2

    | x  1½   . |
R = |-1½  x   0 |
    | .   0   x |
```

## Example Provinciaal Brabants Kampioenschap PNDB 2018

| Pl | | Name | rating | R1 | R2 | R3 | R4 | R5 | R6 | R7 | R8 | + | ± | N | % ± | It1 | It2 | It3 | It4 | It5 | It6 | Rk |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | mi | Frank Teer | 2208 | 2/3b | 2/13w | 1/2b | 2/4w | 2/5b | 2/17w | 2/6w | 2/7b | 7 | 7 | 8 | 43,8% | 54,3% | 60,0% | 61,1% | 62,3% | 62,4% | 62,8% | 2 |
| 2 | mf | Andrew Tjon A Ong | 2192 | 2/21w | 2/6b | 1/1w | 2/7w | 2/12b | 2/4w | 2/9b | 2/8w | 7 | 7 | 8 | 43,8% | 54,7% | 59,9% | 61,6% | 62,7% | 63,0% | 63,3% | 1 |
| 3 | | Jan Kornilov | | 0/1w | 1/11b | 1/15b | 2/16w | 1/23w | 2/14b | | 1/4b | 2 | 1 | 7 | 7,1% | 12,2% | 11,4% | 12,2% | 11,7% | 11,9% | 11,7% | 9 |
| 4 | | Piet van Erp | | | 2/21b | 1/8w | 0/1b | 2/10w | 0/2b | 2/17b | 1/3w | 3 | 1 | 7 | 7,1% | 20,7% | 24,1% | 26,6% | 27,2% | 27,9% | 28,0% | 3 |
| 5 | | Luud Ector | 1908 | 2/19w | 1/12b | 1/18w | 2/8b | 0/1w | 0/6b | 2/22w | 2/15b | 4 | 2 | 8 | 12,5% | 17,3% | 18,9% | 19,8% | 20,0% | 20,2% | 20,2% | 6 |
| 6 | | Wiebe Cnossen | 1916 | 1/20w | 0/2w | 1/11b | 2/19b | 2/15w | 2/5w | 0/1b | 1/14b | 3 | 1 | 8 | 6,3% | 16,9% | 15,6% | 16,9% | 16,2% | 16,5% | 16,3% | 8 |
| 7 | | Jules Martens | 2001 | 2/17b | 2/15w | 1/10w | 0/2b | 1/22b | 1/8b | 2/21w | 0/1w | 3 | 1 | 8 | 6,3% | 14,7% | 17,0% | 19,7% | 20,1% | 20,8% | 20,9% | 4 |
| 8 | | Ties Slagter | 1955 | 2/22b | 2/23w | 1/4b | 0/5w | 1/9b | 1/7w | 2/10w | 0/2b | 3 | 1 | 8 | 6,3% | 12,1% | 17,1% | 18,6% | 19,8% | 20,2% | 20,5% | 5 |
| 9 | | Jan van den Hooff | 1996 | 1/16w | 0/10b | 2/19w | 2/23b | 1/8w | 1/12w | 0/2w | 2/22b | 3 | 1 | 8 | 6,3% | 8,3% | 7,9% | 8,8% | 8,6% | 9,0% | 8,9% | 11 |
| 10 | | Ton Sprangers | | 2/11w | 2/9w | 1/7b | 0/12b | 0/4b | 2/24w | 0/8b | 2/18w | 4 | 1 | 8 | 6,3% | 8,3% | 9,9% | 10,1% | 10,8% | 10,8% | 11,0% | 10 |
| 11 | | Lev Gilevych | | 0/10b | 1/3w | 1/6w | 2/21b | 1/14w | 1/22b | 1/18w | 2/17b | 2 | 1 | 8 | 6,3% | 3,8% | 4,7% | 4,3% | 4,5% | 4,4% | 4,5% | 12 |
| 12 | | Frank Swagemakers | 1900 | 2/25w | 1/5w | 2/20b | 2/10w | 0/2w | 1/9b | | | 3 | 2 | 6 | 16,7% | 18,3% | 18,7% | 18,6% | 18,6% | 18,6% | 18,6% | 7 |
| 13 | | Joop Achterstraat | 1993 | 2/14b | 0/1b | 1/16w | 2/20w | 0/17b | 1/18b | 0/15b | 2/23w | 3 | 0 | 8 | 0,0% | -0,5% | -3,9% | -4,1% | -5,1% | -5,1% | -5,4% | 15 |
| 14 | | Tanya-Marie Cnossen | 1942 | 0/13w | 2/16b | 1/23b | 1/22w | 1/11b | 0/3w | 2/20b | 1/6w | 2 | 0 | 8 | 0,0% | -4,7% | -5,8% | -7,1% | -7,6% | -7,9% | -8,1% | 16 |
| 15 | | Martien van Erp | | 1/18b | 0/7b | 1/3w | 2/25w | 0/6b | 2/23b | 2/13w | 0/5w | 3 | 0 | 8 | 0,0% | -4,8% | -2,8% | -4,5% | -3,9% | -4,4% | -4,2% | 14 |
| 16 | | Arnold Beset | | 1/9b | 0/14w | 1/13b | 0/3b | 0/24w | | 2/25w | 2/21b | 2 | -1 | 7 | -7,1% | -18,0% | -21,2% | -24,0% | -24,4% | -25,2% | -25,1% | 20 |
| 17 | | Roland Coray | | 0/7w | 2/25b | 1/22b | 2/18b | 2/13w | 0/1b | 0/4w | 0/11w | 3 | -1 | 8 | -6,3% | -7,1% | -4,2% | -4,2% | -3,3% | -3,4% | -3,1% | 13 |
| 18 | | Yaroslav Gilevych | | 1/15w | 1/19b | 1/5b | 0/17w | 2/20b | 1/13w | 1/11b | 0/10b | 1 | -1 | 8 | -6,3% | -6,6% | -9,9% | -9,6% | -10,5% | -10,3% | -10,6% | 17 |
| 19 | | Piet Jonkers | 1817 | 0/5b | 1/18w | 0/9b | 0/6w | | 0/21b | 2/24b | 2/25w | 2 | -2 | 7 | -14,3% | -24,4% | -25,4% | -27,9% | -27,8% | -28,5% | -28,3% | 22 |
| 20 | | Oleksandra Chumachenko | 1914 | 1/6b | 2/24w | 0/12w | 0/13b | 0/18w | 2/25b | 0/14w | | 2 | -2 | 7 | -14,3% | -22,9% | -24,7% | -27,9% | -28,1% | -29,0% | -29,0% | 23 |
| 21 | | Johan Rijnen | | 0/2b | 0/4w | 2/24b | 0/11w | 2/25b | 2/19w | 0/7b | 0/16w | 3 | -2 | 8 | -12,5% | -16,7% | -18,1% | -18,8% | -18,9% | -19,0% | -19,0% | 19 |
| 22 | | Harm van der Veen | | 0/8w | | 1/17w | 1/14b | 1/7w | 1/11w | 0/5b | 0/9w | 0 | -3 | 7 | -21,4% | -16,0% | -15,0% | -13,5% | -12,9% | -12,6% | -12,4% | 18 |
| 23 | | Simon Rompa | 1833 | 2/24b | 0/8b | 1/14w | 0/9w | 1/3b | 0/15w | | 0/13b | 1 | -3 | 7 | -21,4% | -22,5% | -24,2% | -25,0% | -25,2% | -25,5% | -25,5% | 21 |
| 24 | | Egor Kornilov | | 0/23w | 0/20b | 0/21w | | 2/16b | 0/10b | 0/19w | | 1 | -4 | 6 | -33,3% | -43,0% | -49,2% | -50,6% | -52,2% | -52,3% | -52,8% | 24 |
| 25 | | Peter van Poppel | | 0/12b | 0/17w | | 0/15b | 0/21w | 0/20w | 0/16b | 0/19b | 0 | -7 | 7 | -50,0% | -54,5% | -60,7% | -61,1% | -62,6% | -62,5% | -63,0% | 25 |

It is easy to check the result of the above calculation as follows:

| | | | |
|---|---|---|---|
| Andrew Tjon A Ong | 0,633 | Frank Teer | 0,627 |
| Opponents | Rb-Rtg | Opponents | Rb-Rtg |
| Johan Rijnen | -0,1903 | Jan Kornilov | 0,1168 |
| Wiebe Cnossen | 0,1629 | Joop Achterstraat | -0,0541 |
| Frank Teer | 0,6275 | Andrew Tjon A Ong | 0,6330 |
| Jules Martens | 0,2088 | Piet van Erp | 0,2798 |
| Frank Swagemakers | 0,1857 | Luud Ector | 0,2024 |
| Piet van Erp | 0,2798 | Roland Coray | -0,0311 |
| Jan van den Hooff | 0,0885 | Wiebe Cnossen | 0,1629 |
| Ties Slagter | 0,2046 | Jules Martens | 0,2088 |
| RbOpp average | 0,1959 | RbAvg | 0,1898 |
| s | 0,4375 + | s | 0,4375 + |
| RbAvg + s | 0,633 | Own Rb-Rtg | 0,627 |

# Generalized Row Sum Method

## Principle

Row sum scores (s) are calculated as the sum over k of the skew symmetric outcomes $r_{ik}$. The generalized row sum method is an extension of the row sum method. The idea of the generalization is to complete the missing comparisons. Central to the statistical interpretation of the generalized row sum method is the assumption that the expected value $E(r_{ij})$ of a missing comparison is proportional to the difference of the scores $x_i$, $x_j$ of the compared players i and j:

- $x_i = s_i + \Sigma_j E(r_{ij})$, where $r_{ij}$ is missing, $s_i = \Sigma_k r_{ik}$ where $r_{ik}$ is defined and
- $E(r_{ij}) = (x_j - x_i) / \gamma$, for any positive factor γ, see Cheboratev, 1989, Chapter 3 Statistical model, p1105

These relationships form a system of n linear equations in n unknowns $x_i$. For $\gamma = m.n + \varepsilon^{-1}$ the system of equations is equivalent to:

- $x_i = \Sigma_k(r_{ik} + \varepsilon(x_k - x_i + r_{ik} n.m))$, k = 1,…,n

where

- ε is a positive parameter
- ε is well-chosen for given n and m if and only if $\varepsilon^{-1} \geq m(n - 2)$
- $r_{ik} + r_{ki} = 0$, the skew symmetric outcome between players i, k
- number of players n > 2, m = maximal number of games between any two players

Parameter ε is said to be well-chosen if for any outcome matrix $R = (r_{ik})$ the value of its contribution to $x_i$ is nonnegative for a maximal win and nonpositive for a maximal loss (Chebotarev, 1997, p14).

## Calculation

The generalized row sums (x) are a solution of the set of lineair equations:

- $(L + \varepsilon^{-1} I)x = \gamma s$
- $\Sigma x = 0$

where

- $L = (l_{ij})$ is the Laplacian matrix of the matches graph, I is the identity matrix
- $\gamma = m.n + \varepsilon^{-1}$

Note that LS-ratings can be obtained as a limit of the generalized row sum calculation by setting $\varepsilon^{-1} = 0$ and $\gamma = 1$.

## Example Trainingsvierkamp Harm Wiersma Huizum 2005 (Blitz)

```
   n = 4                                             m = 2, 1/ε = 2×4
Pl Naam                      Rating   1  2  3  4  N  |Matches|L + (1/ε)I | + = - Pts  s  |Rk  x    |
 1 Harm Wiersma           GMI  1502  xx 02 22 ..  4  |0 2 2 0|12 -2 -2  0| 3 0 1   6  2  | 1  2,667|
 2 Rein van der Pal       MF   1402  20 xx 02 12  6  |2 0 2 2|-2 14 -2 -2| 3 1 2   7  1  | 2  1,000|
 3 Tjalling van den Bosch      1190  00 20 xx 21  6  |2 2 0 2|-2 -2 14 -2| 2 1 3   5 -1  | 3 -1,000|
 4 Jan Adema                   1217  .. 10 01 xx  4  |0 2 2 0| 0 -2 -2 12| 0 2 2   2 -2  | 4 -2,667|

 x = column vector of generalized row sum ratings
 s = column vector wins -/- losses
```

### Statistical model

```
Check expected values xi == si + Σ(xi-xj)/γ, γ = 1/ε + n.m, γ = 16
All players are expected to complete an equal number of games

  Harm Wiersma      2,67                 Jan Adema         -2,67
  s1                2                    s4                -2
Missing games                         Missing games
  (x1-x4)/γ         0,333                (x4-x1)/γ         -0,333
  (x1-x4)/γ         0,333                (x4-x1)/γ         -0,333
                    ----- +                                ------ +
  s1 + missing gms  2,67                 s1 + missing gms  -2,67
```

### Direct method

```
Check (L + (1/ε)I)x == γs, 1/ε = 8, γ = 16
                   L+(1/ε)I    x       (L + (1/ε)I)x
 Harm Wiersma            12   2,6667  32,0004
 Rein van der Pal        -2   1,0000  -2,0000
 Tjalling van den Bosch  -2  -1,0000   2,0000
 Jan Adema                0  -2,6667   0,0000
                                       ------ +
 γ.s1                                 32,0000
 s1                                    2,000
```

- The statistical model is useful when the tournament is nearly complete
- For sparse tournament matrices, the direct method is more practical

# References

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